Chapter 1 - Degrees and Subgaphs

Let $G = (V, E)$ be a graph.

- For $v \in V$, the se to neighbors is denoted as $N_G(v) = N(v) = \{u : uv \in E\}$.
- For $v \in V$ the **degree** of v, is

$$deg_G(v) = deg(v) = d_G(v) = d(v) = |N(v)|.$$

- An **isolated vertex** is a vertex with degree 0.
- A **leaf** is a vertex with degree 1.
- The **minimum degree** of G, is $\delta(G) := \min\{d(v) : v \in V)\}$.
- The **maximum degree** of G is $\Delta(G) := \max\{d(v) : v \in V\}$.
- The **average degree** degree of a graph G is

$$d(G) := \frac{1}{|V|} \sum_{v \in V} d(v).$$

  Notice that $\delta(G) \leq d(G) \leq \Delta(G)$.

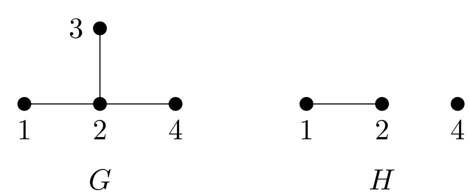
- A graph is called k**-regular** if $\delta(G) = \Delta(G) = k$, i.e. every vertex has degree k. (if $k = 3$, we call G cubic).

1: Show that for any graph $G = (V, E)$ holds

- $\sum_{v \in V} d(v) = 2|E|$
- The number of vertices of odd degree is always even.

Solution: The first item is an obvious double-counting argument.
If the number of odd degree vertices is odd, then $\sum_{v\in} d(v)$ is odd, which contradicts it being equal to $2|E|$.

Graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, notation $H \subseteq G$.



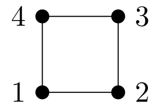
H is a **proper subgraph** if $H \subseteq G$, $H \neq G$ (and H is not a null graph).

H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$

H is a **induced subgraph** if $H \subseteq G$ and $\forall u, v \in V(H), uv \in E(G) \Rightarrow uv \in E(H)$.

If $X \subseteq V(G)$, then $G[X]$ denotes induced subgraph H of G where $V(H) = X$.

2: Count the number of subgraphs, spanning subgraphs, and induced subgraphs of C_4.



C_4

Solution: Spanning: 2^4 by deciding for each edge if it is staying or not
Induced: 2^4 by deciding for each vertex if it is staying or not
Subgraphs: By number of vertices. 4 vertices give 2^4 subgraphs. 3 vertices give 4 subgraphs each and there are 4 of them, so 4×4. On 2 vertices, the diagonal are just 2 graphs. For the 4 edges, each counts as 2 subgraphs. In total on 2 vertices, there are 10 of them. On 1 vertex, there are 4. On 0 vertices just 1. In total, $16+16+10+4+1 = 47$

3: Let G be a graph with at least one edge. Let $\varepsilon(G) := \frac{1}{2}d(G) \geq \frac{1}{2}\delta(G)$. Show that there exists $H \subseteq G$, where $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.
"If graph has large average degree, it has a subgraph of large minimum degree."

Solution: Keep deleting from G vertices of degree less or equal to $\varepsilon(G)$. Notice that the average degree is not decreasing by doing this. If we cannot delete any more vertices, we get to $\delta(H) > \varepsilon(H)$. Note that H is not empty, because before empty would be K_1 and $\varepsilon(K_1) = 0 < \varepsilon(G)$.

We use $-$ to denote removing edges or vertices to graph, for example $G - v$ or $G - e$. Do not use \.

A **walk** in a graph G is a sequence $v_1, e_1, v_2, e_2, v_3, \ldots, v_n$, where $v_i \in V(G)$ and $e_i \in E(G)$, where consecutive entries are incident.

A **trail** in a graph G is a walk without edge repetition.

A **path** in a graph G is a walk without any repetition.

A **cycle** in a graph G is a sequence $v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n, v_1$, where $v_i \in V(G)$ and $e_i \in E(G)$, where consecutive entries are incident, and has no repetitions except v_1.

A **length** of the above is number of edges.

A **distance** of u and v in G is the length of the shortest path with endvertices u and v. Demote by $dist(u, v)$ or $d(u, v)$.

4: Show that every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.

Solution: A) Take the longest path.
B) Take the longest path and look at neighbors of one of the end vertices.

A **diameter** of a graph G, denote by $diam(G)$ is the length of the longest shortest path.

$$diam(G) = \max_{u,v \in V} dist(u, v)$$

A **girth** of a graph G is the length of the shortest cycle.

A **circumference** of a graph G is the length of the longest cycle.

5: Show that if G contains a cycle, then $g(G) \leq 2diam(G) + 1$.

Solution: If $g(G) \geq 2diam(G)+2$, we find 2 vertices in distance $diam(G)+1$ that do not have a shorter path between them. Opposite vertices on the cycle will do. Shorter path would violate the girth condition.

A vertex is **central** of a graph $G = (V, E)$ its greatest distance from any other vertex is as small as possible. The distance is called **radius**.

$$rad(G) = \min_{v \in V} \max_{y \in V} d(v, y),$$

where $d(v, y)$ is the distance between v and y.

6: Show that

$$rad(G) \leq diam(G) \leq 2rad(G).$$

Solution: Since $diam(G) = \max_{y \in V} d(v, y)$, the first inequality is obvious. Let P be a path in G if length $diam(G)$ and v be its middle vertex. Then distance from v to each of the endpoints is upper bounded by $rad(G)$. Twice that is $2rad(G)$.

7: Show that a graph with radius r and maximum degree $d \geq 3$ has at most $\frac{d}{d-2}(d-1)^r$ vertices.

Solution: Let v be a central vertex. Let D_i be vertices at distance i from v. Notice that $0 \leq i \leq r$. Next see that $|D_i| \leq |D_{i-1}|(d-1)$.

$$|V(G)| \leq 1 + d\sum_{i=0}^{r}(d-1)^i = 1 + \frac{d}{d-2}\left((d-1)^r - 1\right) < \frac{d}{d-2}(d-1)^r$$

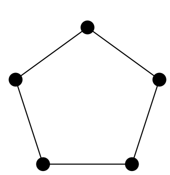
The **complement** $\overline{G}$ of a graph G is graph where $V(\overline{G}) = V(G)$ and $uv \in E(G)$ iff $uv \notin E(\overline{G})$.

Let $G = (V, E)$ be a graph. A **line graph** of G denoted by

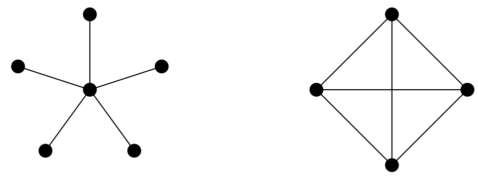
$$L(G) = (E, \{\{e_1, e_2\} : |e_1 \cap u_2| = 1\})$$

is graph on edges where two are adjacent of they share a common neighbor.

8: Find a line graph of the following graph C_5.



9: Use the step by step proof to show that the following graph on 10 vertices has a subgraph of above-average degree. Do not just find it, but follow the proof of the respective theorem. See how is the average degree changing in each step.



Solution: